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On the leading terms ideals of polynomial ideals over a valuation ring

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ABSTRACT

We construct an example of a finitely generated ideal I of $\mathbf{V}[X]$, where $\mathbf{V}$ is a one-dimensional valuation ring, whose leading terms ideal is not finitely generated. This gives a negative answer to the open question of whether if $\mathbf{V}$ is a valuation ring with Krull dimension $\leqslant 1$, then for any finitely generated ideal I of $\mathbf{V}[X]$, the leading terms ideal of I is also finitely generated. The valuation rings satisfying this latter property will be called 1-Gröbner and are studied in this paper.

Introduction

Recall that according to [7] a ring $\mathbf{R}$ is said to be *Gröbner* if for every $n \in \mathbb{N}$ and every finitely generated ideal I of $\mathbf{R}[X_1, \ldots, X_n]$, fixing a monomial order on $\mathbf{R}[X_1, \ldots, X_n]$, the ideal $\mathrm{LT}(I)$ generated by the leading terms of the elements of I is finitely generated. The *Gröbner ring conjecture* [7] says that a valuation ring is Gröbner if and only if its Krull dimension is $\leqslant 1$. A partial solution "if a valuation domain $\mathbf{V}$ is Gröbner then $\dim \mathbf{V} \leqslant 1$" to this conjecture was given in [3].

In [6], it is proved that a valuation domain $\mathbf{V}$ satisfies the property "for any finitely generated ideal I of $\mathbf{V}[X]$ the ideal $\mathrm{LT}(I)$ is finitely generated" if and only if its Krull dimension is $\leqslant 1$. This

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proved the Gröbner ring conjecture in one variable, and also gives the first example of a class of non-Noetherian rings (of Krull dimension 1) satisfying the property above. This result raised the following question [6, Question 1]:

Question. Is true that if $\mathbf{V}$ is a valuation ring (i.e., a ring in which every two elements are comparable under division) with zero-divisors of Krull dimension $\leqslant 1$, then for any finitely generated ideal I of $\mathbf{V}[X]$, the leading terms ideal $\mathrm{LT}(I)$ is also finitely generated?

In this paper, we give a negative answer to this question by constructing a counterexample. The rings $\mathbf{R}$ satisfying the property that "for any finitely generated ideal I of $\mathbf{R}[X]$, the leading terms ideal $\mathrm{LT}(I)$ is also finitely generated" will be called 1-Gröbner rings and are studied in this paper.

1. The 1-Gröbner property and annihilators

Notation 1. Let $\mathbf{R}$ be a commutative ring. For a polynomial $g \in \mathbf{R}[X_1, \dots, X_k]$, we denote by $\mathrm{LT}(g)$ the leading term of g and by $\mathrm{LC}(g)$ its leading coefficient (accordingly to a fixed monomial order).

For $n \in \mathbb{N}$ and I an ideal of $\mathbf{R}[X]$, we denote by $\mathrm{LC}_n(I)$ the ideal of $\mathbf{R}$ generated by the leading coefficients of the elements of I of degree n. In particular, $\mathrm{LC}_0(I) = I \cap \mathbf{R}$. The sequence $(\mathrm{LC}_n(I))_{n\in\mathbb{N}}$ is obviously nondecreasing and so $\mathrm{LC}_\infty(I) := \bigcup_{n\in\mathbb{N}} \mathrm{LC}_n(I)$ is an ideal of $\mathbf{R}$.

In [6], the following theorem was proved.

Theorem 2. *(See [6, Theorem 4].) For a valuation domain* $\mathbf{V}$, *the following assertions are equivalent*:

1. *For any finitely generated ideal I of $\mathbf{V}[X]$, the ideal $\mathrm{LT}(I)$ is also finitely generated.*
2. *If J is a finitely generated ideal of $\mathbf{V}[X]$, then $J \cap \mathbf{V}$ is a principal ideal of $\mathbf{V}$.*
3. $\dim \mathbf{V} \leqslant 1$.

Our goal is to generalize the result above to valuation rings (with zero-divisors). Recall that a ring $\mathbf{V}$ is called a valuation ring if for all $a, b \in \mathbf{V}$, either a divides b or b divides a. The ring $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime number and $k \geqslant 2$, is an example of a valuation ring which is not a domain. To lighten the notation, we give the following definition.

Definition 3. We say that a ring $\mathbf{R}$ is 1-*Gröbner* if for any finitely generated ideal I of $\mathbf{R}[X]$, the ideal $\mathrm{LT}(I)$ is also finitely generated.

The following lemma is immediate and well known.

Lemma 4. *Let $\mathbf{R}$ be a ring. A term aX^k (where $a \in \mathbf{R}$ and $k \in \mathbb{N}$) belongs to an ideal of $\mathbf{R}[X]$ of the form $\langle b_\lambda X^{k_\lambda};\ \lambda \in \Lambda\rangle$, where $b_\lambda \in \mathbf{R}$ and $k_\lambda \in \mathbb{N}$, if and only if $a \in \langle b_\lambda;\ k_\lambda \leqslant k\rangle$.*

As an immediate consequence, one obtains:

Corollary 5. *For any ring $\mathbf{R}$, and any ideal I of $\mathbf{R}[X]$, if $\mathrm{LT}(I)$ is finitely generated, then for all $n \in \mathbb{N} \cup \{\infty\}$, $\mathrm{LC}_n(I)$ is a finitely generated ideal of $\mathbf{R}$.*

Proof. Denoting by $\mathrm{LT}(I) = \langle b_i X^{k_i}\colon\ 1 \leqslant i \leqslant s\rangle$, where $b_i \in \mathbf{R}$ and $k_i \in \mathbb{N}$, by virtue of Lemma 4, we have

$$\mathrm{LC}_n(I) = \langle b_i\colon\ k_i \leqslant n\rangle. \qquad \square$$

Notation 6. Let a be an element of a ring $\mathbf{R}$. Recall that the annihilator of a in $\mathbf{R}$ is the ideal

$$\mathrm{Ann}(a) := \{x \in \mathbf{R} \mid xa = 0\}.$$

As the sequence $(\mathrm{Ann}(a^n))_{n\in\mathbb{N}}$ is nondecreasing,

$$\mathrm{Ann}\big(a^\infty\big) := \bigcup_{n\in\mathbb{N}} \mathrm{Ann}\big(a^n\big)$$

is an ideal of $\mathbf{R}$.

For example, if a is regular then $\mathrm{Ann}(a^\infty) = \{0\}$, and if it is nilpotent $\mathrm{Ann}(a^\infty) = \mathbf{R}$.

Lemma 7. *Let* $\mathbf{R}$ *be a ring. For any* $a \in \mathbf{R}$, *we have*

$$\langle 1 + aX\rangle \cap \mathbf{R} = \mathrm{Ann}\big(a^\infty\big) \quad \textit{and} \quad \mathrm{LT}\big(\langle 1 + aX\rangle\big) = \mathrm{Ann}\big(a^\infty\big)[X] + \langle aX\rangle.$$

In particular, $\mathrm{LT}(\langle 1 + aX\rangle)$ *is finitely generated if and only if so is* $\mathrm{Ann}(a^\infty)$.

Proof. Letting $c \in \langle 1 + aX\rangle \cap \mathbf{R}$, there exists $g = \sum_{i=0}^m b_i X^i \in \mathbf{R}[X]$ such that

$$(1 + aX)g = c \in \mathbf{R}.$$

By identification, we have $ab_m = 0$, $b_m + ab_{m-1} = 0, \ldots, b_1 + ab_0 = 0$, $b_0 = c$, and thus $b_k = (-a)^k c$, $\forall 0 \leqslant k \leqslant m$ and $a^{m+1}c = 0$.

Conversely, letting $b \in \mathrm{Ann}(a^\infty)$, there exists $n \in \mathbb{N}$ such that $ba^n = 0$. It follows that

$$b(1 + aX)\big(1 - aX + \cdots + (-a)^{n-1}X^{n-1}\big) = b\big(1 - (-a)^n X^n\big) = b,$$

and thus $b \in \langle 1 + aX\rangle \cap \mathbf{R}$. We conclude that $\langle 1 + aX\rangle \cap \mathbf{R} = \mathrm{Ann}(a^\infty)$ and necessarily $\mathrm{Ann}(a^\infty)[X] + \langle aX\rangle \subseteq \mathrm{LT}(\langle 1 + aX\rangle)$.

Letting $f = c_0 + c_1X + \cdots + c_nX^n \in \langle 1 + aX\rangle$ (we suppose that $n \geqslant 1$), there exists $g = \sum_{i=0}^m b_i X^i \in \mathbf{R}[X]$ $(m + 1 \geqslant n)$ such that

$$(1 + aX)g = f.$$

By identification, we have

$$S\colon \begin{cases} ab_m = 0, \\ b_m + ab_{m-1} = 0, \\ \vdots \\ b_{n+1} + ab_n = 0, \\ b_n + ab_{n-1} = c_n, \\ \vdots \\ b_1 + ab_0 = c_1, \\ b_0 = c_0, \end{cases}$$

and thus $b_n = c_n - ac_{n-1} + \cdots + (-a)^n c_0$ and $a^{m-n+1} b_n = 0$. It follows that $b_n \in \mathrm{Ann}(a^\infty)$ and $c_n \in \mathrm{Ann}(a^\infty) + \langle a \rangle$, as desired.

The final particular affirmation easily follows by adapting the second members in the equalities of S. $\square$

Proposition 8. *For any ring* $\mathbf{R}$*, we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *where*:

(i) $\mathbf{R}$ *is* 1*-Gröbner.*
(ii) *If* J *is a finitely generated ideal of* $\mathbf{R}[X]$*, then* $J \cap \mathbf{R}$ *is a finitely generated ideal of* $\mathbf{R}$*.*
(iii) *For any* $a \in \mathbf{R}$*,* $\mathrm{Ann}(a^\infty)$ *is a finitely generated ideal of* $\mathbf{R}$*.*

Proof. "(i) $\Rightarrow$ (ii)" This follows from Corollary 5 by taking $n = 0$.

"(ii) $\Rightarrow$ (iii)" This follows immediately from Lemma 7. $\square$

In the following, we give an example of a ring in which assertion (iii) of Proposition 8 fails. It is easy to see that the property "1-Gröbner" is inherited by localization. In opposition to this, the following example shows also that if $\mathbf{R}$ is 1-Gröbner, and $\mathfrak{a}$ is an ideal of $\mathbf{R}$, then $\mathbf{R}/\mathfrak{a}$ need not be 1-Gröbner.

Example 9. Take $X_0, X_1, X_2, \ldots$ infinitely many independent indeterminates over a field $\mathbf{K}$ and consider the ring $\mathbf{R} := \mathbf{K}[X_n\colon\ n \geqslant 0]/\langle X_0^k X_k\colon\ k \geqslant 1\rangle$. Then, clearly

$$\mathrm{Ann}\big(\bar{X}_0^\infty\big) = \langle \bar{X}_k\colon\ k \geqslant 1\rangle,$$

which is not a finitely generated ideal of $\mathbf{R}$. It follows, by virtue of Proposition 8, that $\mathbf{R}$ is not 1-Gröbner, though $\mathbf{K}[X_n\colon\ n \geqslant 0]$ is 1-Gröbner because a finitely generated ideal of $\mathbf{K}[X_n\colon\ n \geqslant 0][X]$ involves in its generators only a finite number of indeterminates among the X_i's.

2. The archimedean property

Recall that a ring $\mathbf{R}$ has dimension $\leqslant 0$ if and only if

$$\forall a \in \mathbf{R},\ \exists n \in \mathbb{N},\ \exists x \in \mathbf{R} \mid a^n = xa^{n+1}. \tag{1}$$

Recall also that a ring $\mathbf{R}$ has Krull dimension $\leqslant 1$ if and only if

$$\forall a, b \in \mathbf{R},\ \exists n \in \mathbb{N},\ \exists x, y \in \mathbf{R} \mid a^n\big(b^n(1 + xb) + ya\big) = 0 \tag{2}$$

or equivalently

$$\forall a, b \in \mathbf{R},\ \exists n \in \mathbb{N} \mid a^n b^n \in a^n b^{n+1}\mathbf{R} + a^{n+1}\mathbf{R}. \tag{3}$$

This is a constructive substitute for the classical abstract definition (see [1,4,5]).

Definition 10. We say that a valuation ring $\mathbf{V}$ is *archimedean* if

$$\forall a, b \in \mathrm{Rad}(\mathbf{V}) \setminus \{0\},\ \exists n \in \mathbb{N} \mid a \text{ divides } b^n,$$

where $\mathrm{Rad}(\mathbf{V})$ denotes the Jacobson radical of $\mathbf{V}$, or also its unique maximal ideal.

For valuation domains, the situation is clear (this is folklore):

Proposition 11. *For any valuation domain* $\mathbf{V}$*, the following three assertions are equivalent*:

(i) $\mathbf{V}$ *is archimedean.*
(ii) *The valuation group of* $\mathbf{V}$ *is archimedean.*
(iii) $\dim \mathbf{V} \leqslant 1$.

For a valuation ring with zero-divisors, the implication "(iii) $\Rightarrow$ (i)" in Proposition 11 is no longer true. The following proposition (suggested by the anonymous referee) gives a characterization of archimedean valuation rings by means of Krull dimension.

Proposition 12. *Let* $\mathbf{V}$ *be a valuation ring. Then,* $\mathbf{V}$ *is archimedean if and only if either* $\dim \mathbf{V} = 0$*, or* $\dim \mathbf{V} = 1$ *and* $\mathbf{V}$ *is integral.*

Proof. Denote by $\mathfrak{m}$ the maximal ideal of $\mathbf{V}$. Assume that $\mathbf{V}$ is archimedean, let $\mathfrak{p}$ be any prime ideal of $\mathbf{V}$ and fix a nonzero element a of $\mathfrak{p}$. Since, for every $b \in \mathfrak{m}$, there exists n such that a divides b^n, $b \in \mathfrak{p}$, and hence, $\mathfrak{m} = \mathfrak{p}$. Conversely, if $\dim \mathbf{V} = 0$, every element of $\mathfrak{m}$ is nilpotent and $\mathbf{V}$ is archimedean.

It is worth pointing out, that the proof above can be transformed into a constructive one (i.e., without using prime ideals) as follows: assume that $\mathbf{V}$ is archimedean. If $\mathbf{V}$ is reduced then it is necessarily integral (for $a, b, c \in \mathbf{V}$, $ab = 0$ and $b = ac \Rightarrow a^2c = 0 \Rightarrow (ac)^2 = 0 \Rightarrow ac = 0 \Rightarrow b = 0$) and thus $\dim \mathbf{V} \leqslant 1$. Otherwise, there exists a nonzero nilpotent element a in $\mathbf{V}$ and hence, as above, any element in $\mathrm{Rad}(\mathbf{V})$ is nilpotent. Thus, $\dim \mathbf{V} = 0$. $\square$

Theorem 13. *For any valuation ring* $\mathbf{V}$*, we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *where*:

(i) $\mathbf{V}$ *is* 1*-Gröbner.*
(ii) *If* J *is a finitely generated ideal of* $\mathbf{V}[X]$*, then* $J \cap \mathbf{V}$ *is a principal ideal of* $\mathbf{V}$*.*
(iii) $\mathbf{V}$ *is archimedean* (*in particular,* $\dim \mathbf{V} \leqslant 1$).

Proof. By virtue of Proposition 8, we have only to prove that "(ii) $\Rightarrow$ (iii)". For this, by way of contradiction, suppose that $\mathbf{V}$ is not archimedean and take $a, b \in \mathrm{Rad}(\mathbf{V}) \setminus \{0\}$ such that b^n divides a for every $n \in \mathbb{N}$. Let us denote by J the ideal of $\mathbf{V}[X]$ generated by $g_1 = bX + 1$ and $g_2 = a$. Because J is finitely generated, $J \cap \mathbf{V}$ is principal, write $J \cap \mathbf{V} = \langle c \rangle$. As $c \in J$, it can be written in the form

$$c = U(X).(bX + 1) + V(X).a,$$

with $U(X), V(X) \in \mathbf{V}[X]$. Denoting by $U(X) = \sum_{i=0}^{k-1} u_i X^i$ and $V(X) = \sum_{j=0}^{k} v_j X^j$, with $u_i, v_j \in \mathbf{V}$ and $k \geqslant \max(1, \deg U + 1, \deg V)$, we have by identification:

$$\begin{aligned} bu_{k-1} + av_k = 0 \quad &\Rightarrow \quad bu_{k-1} = a(-v_k), \\ bu_{k-2} + u_{k-1} + v_{k-1}a = 0 \quad &\Rightarrow \quad b^2u_{k-2} = a(v_k - bv_{k-1}), \\ &\vdots \\ b^k u_0 = a\gamma, \quad &\text{where } \gamma = \sum_{i=1}^{k} (-1)^{k-i} b^{k-i} v_i. \end{aligned}$$

Now, $c = u_0 + v_0 a \Rightarrow b^k c = b^k u_0 + b^k v_0 a = a(\gamma + b^k v_0) = ar$ where $r = \gamma + b^k v_0 \in \mathbf{V}$.

On the other hand, let $x \in \mathbf{V}$ be such that $a = xb^{k+1}$. For $1 \leqslant j \leqslant k+1$ let $x_j = xb^{k+1-j}$, so that $x_j b^j = a$. We have $x_1 g_1 - X g_2 = x_1 =: g_3 \in J, \ldots, g_{k+2} := x_k \in J$, $g_{k+3} := x_{k+1} = x = x_{k+1} g_1 - X g_{k+2} \in J$. Thus, c divides x_{k+1}, and then cb^{k+1} divides a. It follows that $a = cb^{k+1}s = arsb$ (for some $s \in \mathbf{V}$) and thus $(1 - rsb)a = 0$. As $1 - rsb \in \mathbf{V}^{\times}$, we infer that $a = 0$, a contradiction. $\square$

3. A counterexample and a new conjecture

Now we give our main example. It shows that, contrary to the case of valuation domains [6]. If $\mathbf{V}$ is a one-dimensional valuation ring with zero-divisors, there may exist a finitely generated ideal J of $\mathbf{V}[X]$ whose leading terms ideal $\mathrm{LT}(J)$ is not finitely generated, giving a negative answer to the question [6, Question 1] we mentioned in the introduction.

Example 14. Let $\mathbf{T}$ be a rank-two valuation domain and take a nonzero element a in the height-one prime ideal of $\mathbf{T}$. Then $\mathbf{V} := \mathbf{T}/\langle a\rangle$ is a one-dimensional valuation ring which is not archimedean (by virtue of Proposition 12), and hence, not 1-Gröbner.

Theorem 13 encourages us to set the following conjecture (of course, only "(iii) $\Rightarrow$ (i)" is to be proved).

Conjecture 15 *(The archimedean conjecture in one variable). For a valuation ring* $\mathbf{V}$*, the following assertions are equivalent*:

(i) $\mathbf{V}$ *is* 1*-Gröbner.*
(ii) *If* J *is a finitely generated ideal of* $\mathbf{V}[X]$*, then* $J \cap \mathbf{V}$ *is a principal ideal of* $\mathbf{R}$.
(iii) $\mathbf{V}$ *is archimedean.*

4. Buchberger's algorithm for coherent archimedean valuation rings

Before giving our next result, let us remind the algorithm given in [7] which generalizes Buchberger's algorithm to Noetherian valuation rings (this algorithm contains a bug which is now corrected in a corrigendum [8] to this paper). Recall that a valuation ring $\mathbf{V}$ is *coherent* if for any $a \in \mathbf{V}$, $\mathrm{Ann}(a)$ is principal.

Definition 16 *(S-polynomials over valuation rings and division algorithm).* (See [7].) Let $\mathbf{V}$ be a coherent valuation ring, $f_i, f_j \in \mathbf{V}[X_1, \dots, X_n] \setminus \{0\}$ $(i \neq j)$, and $>$ a monomial order.

(i) Denoting by $\mathrm{mdeg}(f_i) = \alpha$ (here, mdeg = multidegree), $\mathrm{mdeg}(f_j) = \beta$, $\gamma = (\gamma_1, \dots, \gamma_n)$, where $\gamma_k = \max(\alpha_k, \beta_k)$ for each k, we define $S(f_i, f_j)$ as follows

$$S(f_i, f_j) = \frac{X^\gamma}{\mathrm{LM}(f_i)} f_i - \frac{\mathrm{LC}(f_i)}{\mathrm{LC}(f_j)} \frac{X^\gamma}{\mathrm{LM}(f_j)} f_j \quad \text{if } \mathrm{LC}(f_j) \text{ divides } \mathrm{LC}(f_i),$$

$$S(f_i, f_j) = \frac{\mathrm{LC}(f_j)}{\mathrm{LC}(f_i)} \frac{X^\gamma}{\mathrm{LM}(f_i)} f_i - \frac{X^\gamma}{\mathrm{LM}(f_j)} f_j$$

$$\text{if } \mathrm{LC}(f_i) \text{ divides } \mathrm{LC}(f_j) \text{ and } \mathrm{LC}(f_j) \text{ does not divide } \mathrm{LC}(f_i).$$

(ii) Denoting $a_i = \mathrm{LC}(f_i)$ and $\mathrm{Ann}(a_i) = \langle d_i\rangle$, $S(f_i, f_i) := d_i f_i$ (it is defined up to a unit).
(iii) As in the classical division algorithm in $\mathbf{F}[X_1, \dots, X_n]$ ($\mathbf{F}$ field) (see [2, p. 61]), for each polynomials $h, h_1, \dots, h_m \in \mathbf{V}[X_1, \dots, X_n]$, there exist $q_1, \dots, q_m, r \in \mathbf{V}[X_1, \dots, X_n]$ such that

$$h = q_1 h_1 + \cdots + q_m h_m + r,$$

where either $r = 0$ or r is a sum of terms none of which is divisible by any of $\mathrm{LT}(h_1), \dots, \mathrm{LT}(h_m)$ (recall that a term aX^α divides a term bX^β, where $a, b \in \mathbf{V}$, if a divides b in $\mathbf{V}$ and X^α divides X^β). The polynomial r is called a remainder of h on division by $H = \{h_1, \dots, h_m\}$ and denoted $r = \bar{h}^H$.

Buchberger's Algorithm for Noetherian valuation rings. *(See [7].) Let* $\mathbf{V}$ *be a Noetherian valuation ring,* $I = \langle g_1, \ldots, g_s \rangle$ *a nonzero ideal of* $\mathbf{V}[X_1, \ldots, X_n]$*, and fix a monomial order* $>$*. Then, a Gröbner basis for* I *can be computed in a finite number of steps by the following algorithm*:

Input: $g_1, \ldots, g_s$
Output: a Gröbner basis G for $\langle g_1, \ldots, g_s \rangle$ with $\{g_1, \ldots, g_s\} \subseteq G$

$G := \{g_1, \ldots, g_s\}$
REPEAT
$G' := G$
For each pair f_i, f_j in G' DO
    $S := \overline{S(f_i, f_j)}^{G'}$
    If $S \neq 0$ THEN $G := G' \cup \{S\}$
UNTIL $G = G'$

The following proposition is a partial answer to Conjecture 15 as we suppose $\mathbf{V}$ to be coherent.

Proposition 17. *Let* $\mathbf{V}$ *be a valuation ring. If* $\mathbf{V}$ *is both coherent and archimedean, then* $\mathbf{V}$ *is* 1*-Gröbner.*

Proof. Let I be a finitely generated ideal of $\mathbf{V}[X]$. A finite basis for $\mathrm{LT}(I)$ can be obtained by executing the generalized version of Buchberger's algorithm over Noetherian valuation rings. In fact, in this algorithm, there is no need of Noetherianity. As a matter of fact, on the one hand, the hypothesis "$\mathbf{V}$ is a valuation ring" is needed for the computation of the S-polynomials of the form $S(f_i, f_j)$ with $i \neq j$, while the coherence hypothesis is needed for the computation of the S-polynomials of the form $S(f_i, f_i)$. Thus, the hypothesis "$\mathbf{V}$ is a coherent valuation ring" ensures the correction of the algorithm. On the other hand, the hypothesis "$\mathbf{V}$ is archimedean" (not all the powers of an element in $\mathrm{Rad}(\mathbf{V}) \setminus \{0\}$ can divide another element in $\mathrm{Rad}(\mathbf{V}) \setminus \{0\}$) ensures its termination (as in the integral case [6]) because it is the same algorithm, only the computation of the $S(f_i, f_i)$ is added. This latter does not affect the termination of the algorithm as $\operatorname{mdeg} S(f_i, f_i) < \operatorname{mdeg}(f_i)$. $\square$

By the following example, we show that an archimedean valuation ring need not be coherent.

Example 18. Let $\mathbf{W}$ be a non-Noetherian valuation domain of Krull dimension 1, denote by $\mathfrak{m}$ its radical (its unique maximal ideal), and consider $\alpha \in \mathfrak{m} \setminus \{0\}$. The ring $\mathbf{V} := \mathbf{W}/\alpha\mathfrak{m}$ is a zero-dimensional (local with $\mathfrak{m}/\alpha\mathfrak{m}$ as radical) valuation ring, and hence, archimedean by virtue of Proposition 12. It is clear that in $\mathbf{V}$, $\mathrm{Ann}(\bar{\alpha}) = \mathfrak{m}/\alpha\mathfrak{m}$ which is not principal as $\mathfrak{m}$ is not principal as an ideal of $\mathbf{W}$.

By virtue of Lemma 7, we know that if a valuation ring $\mathbf{V}$ is 1-Gröbner, then for any $a \in \mathbf{V}$, $\mathrm{Ann}(a^\infty)$ is principal. This raises the following question in connexion with Conjecture 15 and Proposition 17:

Question 19. Is a 1-Gröbner valuation ring coherent?

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